Topology and its Applications 159 (2012) 437–449

Contents lists available at SciVerse ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Solvable group representations and free divisors whose complements are $K(\pi, 1)$'s

James Damon *,[1], Brian Pike

Department of Mathematics, University of North Carolina, Chapel Hill, NC 27599-3250, USA

ARTICLE INFO

MSC:
primary 22E25, 55P20, 55R80
secondary 20G05, 32S30

Keywords:
Solvable linear algebraic groups
Block representations
Exceptional orbit varieties
Linear free divisors
(Modified) Cholesky-type factorizations
Eilenberg–Mac Lane spaces
Relative invariants
Cohomology of complements
Cohomology of Milnor fibers

ABSTRACT

We apply previous results on the representations of solvable linear algebraic groups to construct a new class of free divisors whose complements are $K(\pi, 1)$'s. These free divisors arise as the exceptional orbit varieties for a special class of "block representations" and have the structure of determinantal arrangements.

Among these are the free divisors defined by conditions for the (modified) Cholesky-type factorizations of matrices, which contain the determinantal varieties of singular matrices of various types as components. These complements are proven to be homotopy tori, as are the Milnor fibers of these free divisors. The generators for the complex cohomology of each are given in terms of forms defined using the basic relative invariants of the group representation.

0. Introduction

A classical result of Arnold and Brieskorn [3,4] states that the complement of the discriminant of the versal unfolding of a simple hypersurface singularity is a $K(\pi, 1)$. Deligne [10] showed this result could be placed in the general framework by proving that the complement of an arrangement of reflecting hyperplanes for a Coxeter group is again a $K(\pi, 1)$ (and more generally for simplicial arrangements). A discriminant complement for a simple hypersurface singularity can be obtained as the quotient of the complement of such a hyperplane arrangement by the free action of a finite group, and hence is again a $K(\pi, 1)$.

What the discriminants and Coxeter hyperplane arrangements have in common is that they are free divisors. This notion was introduced by Saito [20], motivated by his discovery that the discriminants for the versal unfoldings of isolated hypersurface singularities are always free divisors. By contrast, Knörrer [15] found an isolated complete intersection singularity, for which the complement of the discriminant of the versal unfolding is not a $K(\pi, 1)$ (although it is again a free divisor by a result of Looijenga [16]).

This leads to an intriguing question about when a free divisor has a complement which is a $K(\pi, 1)$. This remains unsettled for the discriminants of versal unfoldings of isolated hypersurface singularities; this is the classical "$K(\pi, 1)$-Problem". Also, for hyperplane arrangements, there are other families such as arrangements arising from Shephard groups which by Orlik and Solomon [17] satisfy both properties; however, it remains open whether the conjecture of Saito is true that every free arrangement has complement which is a $K(\pi, 1)$. A survey of these results on arrangements can be found in the book of Orlik and Terao [18]. Except for isolated curve singularities in $\mathbb{C}^2$ (and the total space for their

* Corresponding author.
E-mail address: jndamon@math.unc.edu (J. Damon).

[1] Partially supported by the National Science Foundation grant DMS-0706941.

0166-8641/$ – see front matter © 2011 Elsevier B.V. All rights reserved.
doi:10.1016/j.topol.2011.09.018

equisingular deformations), there are no other known examples of free divisors whose complements are $K(\pi,1)$'s. While neither $K(\pi,1)$-problem has been settled, numerous other classes of free divisors have been discovered so this question continues to arise in new contexts.

In this paper, we define a large class of free divisors whose complements are $K(\pi,1)$'s by using the results obtained in [8]. These free divisors are "determinantal arrangements", which are analogous to hyperplane arrangements except that we replace a configuration of hyperplanes by a configuration of determinantal varieties (and the defining equation is a product of determinants rather than a product of linear factors).

These varieties arise as the "exceptional orbit varieties" for representations of solvable linear algebraic groups which are "Block Representations" in the sense of [8]; and in Theorem 3.1 we show that their complements are always $K(\pi,1)$'s, where π is the extension of a finitely generated free abelian group by a finite group. More generally we show that for a weaker notion of "nonreduced Block Representation", the exceptional orbit varieties are weaker free* divisors; however their complements are still $K(\pi,1)$'s. From this we deduce in Theorem 3.2 that the Milnor fibers of these exceptional orbit varieties are again $K(\pi,1)$'s.

We exhibit in Theorem 3.4 a number of families of such "determinantal arrangements" in spaces of symmetric, skew-symmetric and general square matrices and $(m-1)\times m$ matrices which are free divisors (or free* divisors) with complements $K(\pi,1)$'s. We note that the individual determinant varieties in these spaces are neither free divisors nor are their complements $K(\pi,1)$'s. However, the determinant variety can be placed in a larger geometric configuration of determinantal varieties which together form a free divisor whose complement is a $K(\pi,1)$.

For these results we use special representations of solvable algebraic groups involved in various forms of Cholesky-type factorizations or modified Cholesky-type factorizations for symmetric, skew-symmetric and general square complex matrices, and $m\times(m+1)$ general matrices. We describe these factorizations in Section 1. We go on to specifically show in these cases $\pi \simeq \mathbb{Z}^k$ where k is the rank of the corresponding solvable groups (Theorem 3.4), so the complements are homotopy equivalent to k-tori. We further deduce that the Milnor fibers for these cases are homotopy equivalent to $(k-1)$-tori. Furthermore, in Theorem 4.1 and Corollary 4.3, we are able to find explicit generators for the complex cohomology of the complement and of the Minor fibers using forms obtained from the basic relative invariants for the group actions, using results from the theory of prehomogeneous spaces due to Sato and Kimura [21]. We deduce that the Gauss–Manin systems for these determinantal arrangements are trivial. The simple form of these results contrasts with the more difficult situation of linear free divisors for reductive groups considered by Granger et al. [12]

These determinantal arrangements are also used in [9] for determining the vanishing topology of more general matrix singularities based on the various determinantal varieties.

The authors are especially grateful to Shrawan Kumar for his comments and references.

1. Cholesky factorizations, modified Cholesky factorizations, and solvable group representations

In this section we begin by explaining the interest in determinantal arrangements which arises from various forms of Cholesky factorization. Traditionally, it is well known that certain matrices can be put in normal forms after multiplication by appropriate matrices. The basic example is for symmetric matrices, where a nonsingular symmetric matrix A can be diagonalized by composing it with an appropriate invertible matrix B to obtain $B\cdot A\cdot B^T$. The choice of B is highly nonunique. For real matrices, Cholesky factorization gives a unique choice for B provided A satisfies certain determinantal conditions.

More generally, by "Cholesky factorization" we mean a general collection of results for factoring real matrices into products of upper and lower triangular matrices. These factorizations are used to simplify the solution of certain problems in applied linear algebra.

We recall the three fundamental cases (see [11] and [1]). For them, we let $A=(a_{ij})$ denote an $m\times m$ real matrix which may be symmetric, general, or skew-symmetric. We let $A^{(k)}$ denote the $k\times k$ upper left-hand corner submatrix.

Theorem 1.1 *(Forms of Cholesky-type factorization).*

(1) Classical Cholesky factorization: *If A is a positive-definite symmetric matrix with* $\det(A^{(k)})\neq 0$ *for $k=1,\dots,m$, then there exists a unique lower triangular matrix with positive diagonal entries B so that $A=B\cdot B^T$.*
(2) Classical LU decomposition: *If A is a general matrix with* $\det(A^{(k)})\neq 0$ *for $k=1,\dots,m$, then there exist a unique lower triangular matrix B and upper triangular matrix C with diagonal entries $=1$ so that $A=B\cdot C$.*
(3) Skew-symmetric Cholesky factorization (see e.g. [1]): *If A is a skew-symmetric matrix for $m=2\ell$ with* $\det(A^{(2k)})\neq 0$ *for $k=1,\dots,\ell$, then there exists a unique lower block triangular matrix B with 2×2-diagonal blocks of the form* a) *in* (1.1) *so that $A=B\cdot J\cdot B^T$, for J the block diagonal $2\ell\times 2\ell$ skew-symmetric matrix with 2×2-diagonal blocks of the form* b) *in* (1.1). *For $m=2\ell+1$, then there is again a unique factorization except now B has an additional entry of* 1 *in the last diagonal position, and J is replaced by J' which has J as the upper left corner $2\ell\times 2\ell$ submatrix, with remaining entries $=0$.*

$$\text{a)}\quad \begin{pmatrix} r & 0\\ 0 & \pm r\end{pmatrix},\quad r>0,\quad \textit{and}\quad \text{b)}\quad \begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix}. \tag{1.1}$$

We note that in each case the polynomial in the entries of A, $\prod \det(A^{(k)})$ over $1 \leqslant k \leqslant m$ (with k even in the skew-symmetric case), defines a real variety off which there is the appropriate Cholesky factorization defined. This real variety is defined on a space of real matrices and can be viewed as a real determinantal arrangement formed from the real varieties defined by the individual $\det(A^{(k)})$. We turn to the corresponding complex situation and identify such varieties as examples of determinantal varieties which arise as "exceptional orbit varieties" of solvable group actions. This perspective leads to a more general understanding of the determinantal varieties associated to Cholesky factorization.

1.1. Cholesky factorizations and determinantal arrangements

We begin with the notion of determinantal varieties and determinantal arrangements on a complex vector space V.

Definition 1.2. A variety $\mathcal{V} \subset V$ is a *determinantal variety* if $\mathcal{V}$ has a defining equation $p = \det(B)$ where $B = (b_{i,j})$ is a square matrix whose entries are linear functions on V. Then, $X \subset V$ is a *determinantal arrangement* if X has defining equation $p = \prod p_i$ where each p_i is a defining equation for a determinantal variety $\mathcal{V}_i$. Then, $X = \bigcup_i \mathcal{V}_i$.

In the simplest case where the determinants are 1×1 determinants, then we obtain a central hyperplane arrangement.

Remark 1.3. In the definitions of determinantal variety and determinantal arrangements we do not require that the defining equations be reduced. In fact, even for certain of the Cholesky-type factorizations above this need not be true.

We now consider the spaces of $m \times m$ complex matrices which will either be symmetric, general, or skew-symmetric (with m even). In the complex case there are the following analogues of Cholesky factorization (see [8] and [19]).

Theorem 1.4 *(Complex Cholesky-type factorization).*

(1) Complex Cholesky factorization: *If A is a complex symmetric matrix with* $\det(A^{(k)}) \neq 0$ *for $k = 1, \dots, m$, then there exists a lower triangular matrix B, which is unique up to multiplication by a diagonal matrix with diagonal entries ± 1, so that $A = B \cdot B^T$.*
(2) Complex LU decomposition: *If A is a general complex matrix with* $\det(A^{(k)}) \neq 0$ *for $k = 1, \dots, m$, then there exist a unique lower triangular matrix B and a unique upper triangular matrix C which has diagonal entries $= 1$ so that $A = B \cdot C$.*
(3) Complex skew-symmetric Cholesky factorization: *If A is a skew-symmetric matrix for $m = 2\ell$ with* $\det(A^{(2k)}) \neq 0$ *for $k = 1, \dots, \ell$, then there exists a lower block triangular matrix B having the same form as in* (3) *of Theorem* 1.1 *but with complex entries of the same signs in each 2×2 diagonal block* a) *of* (1.1) (*i.e.* $= r \cdot I$), *so that $A = B \cdot J \cdot B^T$, for J the $2\ell \times 2\ell$ skew-symmetric matrix as in* (3) *of Theorem* 1.1. *Then, B is unique up to multiplication by block diagonal matrices with 2×2 diagonal blocks $= \pm I$. There is also a factorization for the case $m = 2\ell + 1$ analogous to that in* (3) *in Theorem* 1.1, *again with complex entries of the same signs in each 2×2 diagonal block.*

The polynomials $\prod \det(A^{(k)})$ over $1 \leqslant k \leqslant m$ (with k even in the skew-symmetric case) define varieties which are determinantal arrangements. However, these varieties have differing properties when viewed from the perspective of their being free divisors. While they are free divisors in the symmetric case, they are a weaker form of free* divisor (see [6]) for the general and skew-symmetric cases. The stronger properties of free divisors discovered in [7] led to a search for a modification of the notion of Cholesky factorization for general $m \times m$ matrices. This further extends to the space of $(m-1) \times m$ general matrices. In each case there is a modified form of Cholesky-type factorization (see [8] and [19]) which we consider next.

For an $m \times m$ matrix A, we let $\hat{A}$ denote the $m \times (m-1)$ matrix obtained by deleting the first column of A. If instead A is an $(m-1) \times m$ matrix, we let $\hat{A}$ denote the $(m-1) \times (m-1)$ matrix obtained by deleting the first column of A. In either case, we let $\hat{A}^{(k)}$ denote the $k \times k$ upper left submatrix of $\hat{A}$, for $1 \leqslant k \leqslant m-1$. Then a modified form of Cholesky factorization is given by the following (see [8] and [19]).

Theorem 1.5 *(Modified Cholesky-type factorization).*

(1) Modified LU decomposition: *If A is a general complex $m \times m$ matrix with* $\det(A^{(k)}) \neq 0$ *for $k = 1, \dots, m$ and* $\det(\hat{A}^{(k)}) \neq 0$ *for $k = 1, \dots, m-1$, then there exist a unique lower triangular matrix B and a unique upper triangular matrix C, which has first diagonal entry $= 1$, and remaining first row entries $= 0$ so that $A = B \cdot K \cdot C$, where K has the form of* a) *in* (1.2).
(2) Modified Cholesky factorization for $(m-1) \times m$ matrices: *If A is an $(m-1) \times m$ complex matrix with* $\det(A^{(k)}) \neq 0$ *for $k = 1, \dots, m-1$,* $\det(\hat{A}^{(k)}) \neq 0$ *for $k = 1, \dots, m-1$, then there exist a unique $(m-1) \times (m-1)$ lower triangular matrix B and*

a unique $m \times m$ matrix C having the same form as in (1)*, so that $A = B \cdot K' \cdot C$, where K' has the form of* b) *in* (1.2)*.*

$$\text{a)} \quad \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \textit{and} \quad \text{b)} \quad \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}. \tag{1.2}$$

This factorization yields unexpected forms and even one for non-square matrices. The corresponding determinantal arrangements are defined by

$$\prod \det\bigl(A^{(k)}\bigr) \cdot \prod \det\bigl(\hat{A}^{(j)}\bigr) = 0$$

where the products are over $1 \leqslant k \leqslant m$, $1 \leqslant j \leqslant m-1$ for case (1) and over $1 \leqslant k \leqslant m-1$ and $1 \leqslant j \leqslant m-1$ for case (2). In the next section, we explain how as a consequence of [8] and [19], these determinantal arrangements are free divisors. As we will explain, these are special cases of a general result which constructs such determinantal arrangements from representations of solvable linear algebraic groups. In fact there are many other families of determinantal arrangements which similarly arise (see [8]). This representation will then allow us to explicitly describe the complements to the determinantal arrangements and give criteria that they are $K(\pi, 1)$'s.

2. Block representations for solvable groups

All of the examples of Cholesky-type factorization given in Section 1 can be viewed as statements about open orbits for representations of solvable linear algebraic groups. For example, for the case of symmetric matrices, there is the representation of the Borel subgroup of $m \times m$ lower triangular matrices B_m acting on the space of $m \times m$ symmetric matrices Sym_m given by

$$C \cdot S = CSC^T \quad \text{for } C \in B_m \text{ and } S \in Sym_m. \tag{2.1}$$

However, not all such representations have the desired properties. We consider a special class of finite dimensional (complex) regular representations $\rho : G \to \mathrm{GL}(V)$ of solvable linear algebraic groups G (throughout this paper the solvable groups will always be understood to be connected). Such a representation will be called an *equidimensional representation* if $\dim G = \dim V$ and $\ker(\rho)$ is finite. We will specifically be interested in the case where G has an open orbit, which is then Zariski open. We refer to the complement, which consists of the orbits of positive codimension, as the *exceptional orbit variety* $\mathcal{E} \subset V$.

Mond first observed that in this situation it may be possible to apply Saito's criterion to conclude that $\mathcal{E}$ is a free divisor. This has led to a new class of "linear free divisors". The question is when does Saito's criterion apply. In the case of reductive groups G, Buchweitz and Mond [5] used quivers of finite type to discover a large collection of linear free divisors. We consider instead the situation for solvable algebraic groups.

Remark 2.1. We note that such representations with a Zariski open orbit were studied many years ago by Sato and Kimura who called them *prehomogeneous vector spaces* except they did not require the representations to be equidimensional. Also, they studied them from the perspective of harmonic analysis (see [21] and [14]).

We consider for a representation $\rho : G \to \mathrm{GL}(V)$ the natural commutative diagram of (Lie) group and Lie algebra homomorphisms (see [8]).

Exponential diagram for a representation:

$$\begin{array}{ccccc} \mathfrak{g} & \xrightarrow{\tilde{\rho}} & \lambda(V) & \xrightarrow{\tilde{i}} & m \cdot \theta(V) \\ \downarrow{\scriptstyle \exp} & & \downarrow{\scriptstyle \exp} & & \downarrow{\scriptstyle \exp} \\ G & \xrightarrow{\rho} & \mathrm{GL}(V) & \xrightarrow{i} & \mathrm{Diff}(V, 0) \end{array} \tag{2.2}$$

Here $\mathrm{Diff}(V, 0)$ denotes the group of germs of diffeomorphisms; $\theta(V)$ denotes the germs of holomorphic vector fields on $V, 0$, but with Lie bracket the negative of the usual Lie bracket; and $\tilde{i}(A)$ is the vector field which at $v \in V$ has the value $A \cdot v$.

For an equidimensional representation, the composition $\tilde{i} \circ \tilde{\rho}$ is a Lie algebra isomorphism onto its image. The image of a vector $v \in \mathfrak{g}$ will be denoted by ξ_v and called a *representation vector field* associated to v. For a basis $\{v_i,\ i = 1, \ldots, N\}$ of $\mathfrak{g}$, we obtain N representation vector fields $\{\xi_{v_i}\}$.

For a basis $\{w_j,\ j=1,\dots,N\}$ of V, we can represent

$$\xi_{v_i} = \sum_{j=1}^{n} a_{j,i} w_j, \quad i=1,\dots,N, \tag{2.3}$$

where $a_{i,j} \in \mathcal{O}_{V,0}$. We refer to the matrix $A=(a_{i,j})$ as the *coefficient matrix*. Its columns are the coefficient functions for the vector fields. For an equidimensional representation with open orbit, the exceptional orbit variety is defined (possibly with nonreduced structure) by the determinant $\det(A)$, which we refer to as the *coefficient determinant*. As Mond observed, by Saito's Criterion, if the coefficient determinant is a reduced defining equation for $\mathcal{E}$, then $\mathcal{E}$ is a free divisor which is called a *linear free divisor*. We shall use the Lie algebra structure for the case of solvable algebraic groups to obtain linear free divisors.

There is a special class of representations of solvable algebraic groups which we introduced in [8].

Definition 2.2. An equidimensional representation V of a connected linear algebraic group G will be called a *block representation* if:

i) there exists a sequence of G-invariant subspaces

$$V = W_k \supset W_{k-1} \supset \cdots \supset W_1 \supset W_0 = (0);$$

ii) for the induced representation $\rho_j : G \to \mathrm{GL}(V/W_j)$, we let $K_j = \ker(\rho_j)$, then $\dim K_j = \dim W_j$ for all j and the equidimensional action of K_j/K_{j-1} on W_j/W_{j-1} has a relatively open orbit for each j;

iii) the relative coefficient determinants p_j for the representations of K_j/K_{j-1} on W_j/W_{j-1} are all reduced and relatively prime in $\mathcal{O}_{V,0}$.

Remark 2.3. If in the preceding definition, both i) and ii) hold, with the relative coefficient determinants non-zero but possibly nonreduced or not relatively prime in pairs, then we say that it is a *nonreduced block representation*.

The two terms "relative coefficient determinants" and "relatively open orbits" are explained in more detail in [8]. For our purposes here, we can briefly explain their meaning by considering the coefficient matrix. We choose a basis for V and $\mathfrak{g}$ formed from bases for the successive W_j/W_{j-1} and $\mathfrak{k}_j/\mathfrak{k}_{j-1}$, $j=k,k-1,\dots,1$, where $\mathfrak{k}_j$ is the Lie algebra of K_j. We obtain a block triangular matrix coefficient matrix for the corresponding representation vector fields.

Block triangular form:

$$\begin{pmatrix} D_k & 0 & 0 & 0 & 0 \\ * & D_{k-1} & 0 & 0 & 0 \\ * & * & \ddots & 0 & 0 \\ * & * & * & \ddots & 0 \\ * & * & * & * & D_1 \end{pmatrix}. \tag{2.4}$$

The "relative coefficient determinants" are $p_j = \det(D_j)$. These are polynomials defined on V even though they are for the representation W_j/W_{j-1}. Also, the condition for "relative open orbits" is equivalent to p_j not being identically 0 (on V).

Then, block representations give rise to free divisors, and in the nonreduced case to free* divisors (see [8] or [19]).

Theorem 2.4. *Let $\rho : G \to \mathrm{GL}(V)$ be a block representation of a solvable linear algebraic group G, with relative coefficient determinants p_j, $j=1,\dots,k$. Then, the "exceptional orbit variety" $\mathcal{E},0 \subset V,0$ is a linear free divisor with reduced defining equation $\prod_{j=1}^{k} p_j = 0$.*

If instead $\rho : G \to \mathrm{GL}(V)$ is a nonreduced block representation, then $\mathcal{E},0 \subset V,0$ is a linear free divisor and $\prod_{j=1}^{k} p_j = 0$ is a nonreduced defining equation for $(\mathcal{E},0)$.*

In [8] it is shown that all of the determinantal arrangements arising from Cholesky-type factorizations in Section 1 are in fact the exceptional orbit varieties for the equidimensional representations of appropriate solvable linear algebraic groups. There is then the following consequence for these determinantal arrangements.

Theorem 2.5.

i) *The determinantal arrangements arising from the cases of Cholesky-type factorization for complex symmetric matrices and modified Cholesky-type factorization for complex general $m \times m$ and $(m-1) \times m$ matrices are exceptional orbit varieties for the block representations of the corresponding solvable algebraic groups. As such they are free divisors.*

Table 1
Solvable groups and (nonreduced) block representations for (modified) Cholesky-type factorization.

Cholesky-type factorization	Matrix space	Solvable group	Representation
Symmetric matrices	Sym_m	B_m	$B \cdot A = BAB^T$
General matrices	$M_{m,m}$	$B_m \times N_m$	$(B, C) \cdot A = BAC^{-1}$
Skew-symmetric matrices	Sk_m	D_m	$B \cdot A = BAB^T$
Modified Cholesky-type factorization			
General $m \times m$ matrices	$M_{m,m}$	$B_m \times C_m$	$(B, C) \cdot A = BAC^{-1}$
General $(m-1) \times m$ matrices	$M_{m-1,m}$	$B_{m-1} \times C_m$	$(B, C) \cdot A = BAC^{-1}$

ii) *The determinantal arrangements arising from the Cholesky-type factorization for complex general $m \times m$ matrices, and $m \times m$ skew-symmetric matrices (m even) are exceptional orbit varieties for the nonreduced block representations of the corresponding solvable algebraic groups. As such they are free* divisors.*

Proof. We list in Table 1, each type of complex (modified) Cholesky-type factorization, the space of complex matrices, and the solvable group and representation which define the factorization. For this table we use the notation that the spaces of complex $m \times m$ matrices are denoted by: Sym_m for symmetric matrices, $M_{m,m}$ for general $m \times m$ matrices, and Sk_m for skew-symmetric matrices. We also let $M_{m-1,m}$ denote the space of complex $m-1 \times m$ matrices. For the groups we use the notation: B_m for the Borel group of $m \times m$ lower triangular matrices; N_m for the nilpotent group of $m \times m$ upper triangular matrices with 1's on the diagonal; for $m = 2\ell$, D_m denotes the group of lower block triangular matrices, with 2×2 diagonal blocks of the form a) in (1.1) with complex entries of the same sign $r \neq 0$ (i.e. $r \cdot I$); while for $m = 2\ell + 1$, D_m denotes the group of lower block triangular matrices with the first ℓ (2×2)-diagonal blocks as above and the last diagonal element $= 1$; and C_m is subgroup of the $m \times m$ upper triangular matrices with 1 in the first entry and other entries in the first row $= 0$.

These are each either block or nonreduced block representations, as is shown in [8], so that Theorem 2.4 applies. □

Remark 2.6. For the case of skew-symmetric matrices, we have not found a modified form of Cholesky factorization for which the resulting determinantal arrangement is a free divisor. However, by extending the results to representations of nonlinear solvable infinite dimensional Lie algebras, we have found a free divisor which is the analogue of the exceptional orbit variety (again see [8] and [19]).

3. Complements of exceptional orbit varieties for block representations of solvable groups

We next see that for block (or nonreduced block) representations, not only are the exceptional orbit varieties free divisors (resp. free* divisors), but they also share the additional property of having a complement which is a $K(\pi, 1)$.

Theorem 3.1. *Let $\rho : G \to \mathrm{GL}(V)$ be a block representation of a solvable linear algebraic group G whose rank is m. Then, the complement of the exceptional orbit variety, $V \backslash \mathcal{E}$, is a $K(\pi, 1)$ where π is isomorphic to an extension of $\mathbb{Z}^m$ by the finite isotropy subgroup for a generic $v_0 \in V$.*

If instead $\rho : G \to \mathrm{GL}(V)$ is a nonreduced block representation, then although $\mathcal{E}, 0 \subset V, 0$ is only a linear free divisor, the complement $V \backslash \mathcal{E}$ is still a $K(\pi, 1)$ with π as above.*

Proof. Let $\mathcal{U}$ denote the Zariski open orbit of G. We choose $v_0 \in \mathcal{U}$. The map $G \to \mathcal{U}$ sending $g \mapsto g \cdot v_0$ is surjective, as is the corresponding derivative map. By the equidimensionality, the isotropy subgroup $H \subset G$ for v_0 is a 0-dimensional algebraic group, and hence finite. By standard results for Lie groups, the induced mapping $G/H \to \mathcal{U}$ sending $gH \mapsto g \cdot v_0$ is a diffeomorphism. As G is connected, $p : G \to G/H$ is a fiber bundle with finite fiber and connected total space; hence, it is a finite covering space.

Also, by the structure theorem for connected solvable groups, G is the extension of its maximal torus $(\mathbb{C}^*)^m$ by its unipotent radical N. It is a standard result for algebraic groups that the nilpotent group N is a Euclidean group, i.e. the underlying manifold is diffeomorphic to some $\mathbb{C}^k$ (for example, by Corollary 4.8 in [2], it is a subgroup of some upper triangular group in $SL_n(\mathbb{C})$, and then Corollary 1.134 and Theorem 1.127 of [13] yield the result). Hence, G has the homotopy type of its maximal torus, which is a $K(\mathbb{Z}^m, 1)$, where $m = \mathrm{rank}(G)$.

Hence, a simple argument using the homotopy exact sequence for a fibration shows that G/H is also a $K(\pi, 1)$ and by basic results on covering spaces, $H \simeq \pi_1(G/H, \bar{e})/p_*(\pi_1(G, e))$ (for $\bar{e} = e \cdot H$). Thus, G/H is a $K(\pi, 1)$ with π isomorphic to the extension of $\mathbb{Z}^m$ by H. □

As a consequence we are able to describe the Milnor fiber of the nonisolated hypersurface singularity $(\mathcal{E}, 0)$.

Theorem 3.2. *Let $\rho : G \to \mathrm{GL}(V)$ be a (nonreduced) block representation of a solvable linear algebraic group G whose rank is m, with $\mathcal{E}$ the exceptional orbit variety. Then, the Milnor fiber of the nonisolated hypersurface singularity $(\mathcal{E}, 0)$ is a $K(\pi, 1)$, where π satisfies*

$$0 \longrightarrow \pi \longrightarrow \pi_1(V\backslash\mathcal{E}) \longrightarrow \mathbb{Z} \longrightarrow 0 \tag{3.1}$$

with $\pi_1(V\backslash\mathcal{E})$ isomorphic to an extension of $\mathbb{Z}^m$ by the finite isotropy subgroup for a generic $v_0 \in V$.

Proof. First, we observe that if h is the reduced homogeneous defining equation for $\mathcal{E}$, then $h : V\backslash\mathcal{E} \to \mathbb{C}^*$ is a global fibration. This follows using the $\mathbb{C}^*$-action. If h has degree d, we can find an open neighborhood U of 1 in $\mathbb{C}^*$, invariant under inverses, so that the function $f(z) = z^d$ has a well-defined branch of the inverse d-th root function, which we denote by θ. For $w_0 \in \mathbb{C}^*$, there is a neighborhood W of w_0 obtained by applying the $\mathbb{C}^*$-action to U, $z \mapsto z \cdot w_0$.

Then, a local trivialization is given by

$$\begin{aligned} &\psi : W \times h^{-1}(w_0) \to h^{-1}(W) \\ &\bigl(w, (z_1, \ldots, z_n)\bigr) \mapsto \theta(z) \cdot (z_1, \ldots, z_n) \end{aligned} \tag{3.2}$$

where $z = w/w_0$. Then by the homogeneity of h,

$$h\bigl(\theta(z) \cdot z_1, \ldots, \theta(z) \cdot z_n\bigr) = \theta(z)^d h(z_1, \ldots, z_n) = z \cdot w_0 = w.$$

As $\mathbb{C}^*$ is connected all fibers of $h : V\backslash\mathcal{E} \to \mathbb{C}^*$ are diffeomorphic. We next show that the Milnor fiber of $(\mathcal{E}, 0)$ is diffeomorphic to a fiber of this fibration. Given $\varepsilon > 0$ and $\delta > 0$ sufficiently small so that $h^{-1}(B_\delta\backslash\{0\}) \cap B_\varepsilon$ is the Milnor fibration of $(\mathcal{E}, 0)$ and if $w \in B_\delta\backslash\{0\}$, then the fibers $h^{-1}(w)$ are transverse to the ε-sphere S^{2n-1}_ε about 0. We further claim that by the $\mathbb{C}^*$-action, the fibers $h^{-1}(w)$, for $w \in B_\delta\backslash\{0\}$ are transverse to all spheres S^{2n-1}_R for $R > \varepsilon$. If $z \in h^{-1}(w)$ with $\|z\| = R > \varepsilon$, we let $a = \frac{\varepsilon}{R}$ and $z' = a \cdot z$. Then,

$$h\bigl(z'\bigr) = h(a \cdot z) = a^d h(z) = \left(\frac{\varepsilon}{R}\right)^d \cdot w = w'.$$

Thus, $h(z') \in B_\delta\backslash\{0\}$ and $\|z'\| = a \cdot \|z\| = \varepsilon$. Multiplication by a sends S^{2n-1}_R to S^{2n-1}_ε and $h^{-1}(w)$ to $h^{-1}(w')$. Since $h^{-1}(w')$ is transverse to S^{2n-1}_ε at z', and transversality is preserved under diffeomorphisms, we conclude that $h^{-1}(w)$ is transverse to S^{2n-1}_R at z.

Hence, on the fibers $X = h^{-1}(w)\backslash B_\varepsilon$, the function $g(z) = \|z\|$ has no critical points. It then follows using Morse theory that $h^{-1}(w)$ is diffeomorphic to the Milnor fiber $h^{-1}(w) \cap B_\varepsilon$ as claimed.

Finally, it is sufficient to show that a fiber F of the fibration $h : V\backslash\mathcal{E} \to \mathbb{C}^*$ is a $K(\pi, 1)$ with π satisfying (3.1). As F is diffeomorphic to the Milnor fiber of $(\mathcal{E}, 0)$, we can at least conclude by e.g. the Kato–Matsumoto theorem that they are both 0-connected, i.e. path-connected (in the special case of $\dim V = 1$ it is trivially true). Next, by the homotopy exact sequence for the fibration, we have

$$\pi_{j+1}(\mathbb{C}^*) \longrightarrow \pi_j(F) \longrightarrow \pi_j(V\backslash\mathcal{E}) \longrightarrow \pi_j(\mathbb{C}^*) \longrightarrow \pi_{j-1}(F). \tag{3.3}$$

If $j > 1$, then both $\pi_j(V\backslash\mathcal{E}) = 0$, $\pi_{j+1}(\mathbb{C}^*) = 0$; hence, $\pi_j(F) = 0$ for $j > 1$. Thus, F is a $K(\pi, 1)$. Also, as F is path-connected, then the long exact sequence (3.3) with $j = 1$ yields (3.1). □

As a corollary we have an important special case.

Corollary 3.3. *Let $\rho : G \to \mathrm{GL}(V)$ be a (nonreduced) block representation of a solvable linear algebraic group G whose rank is m so that the complement of the exceptional orbit variety $\mathcal{E}$ satisfies $\pi_1(V\backslash\mathcal{E}) \simeq \mathbb{Z}^m$. Then, $V\backslash\mathcal{E}$ is homotopy equivalent to an m-torus, and the Milnor fiber of the nonisolated hypersurface singularity $(\mathcal{E}, 0)$ is homotopy equivalent to an $(m-1)$-torus.*

Proof. By the hypothesis and Theorem 3.1, $V\backslash\mathcal{E}$ is a $K(\mathbb{Z}^m, 1)$ and hence is homotopy equivalent to the m-torus. Second, by Theorem 3.2, the Milnor fiber is a $K(\pi, 1)$ where π is a subgroup of $\mathbb{Z}^m$ with quotient $\mathbb{Z}$. Thus, π is a free abelian group and by comparing ranks, $\pi \simeq \mathbb{Z}^{m-1}$. Thus, the Milnor fiber is homotopy equivalent to an $(m-1)$-torus. □

One example of the usefulness of these theorems is their general applicability to complements of determinantal arrangements in spaces of matrices corresponding to Cholesky or modified Cholesky-type factorizations, as well as to other (nonreduced) block representations given in [8] and [9].

3.1. Determinantal arrangements whose complements are $K(\mathbb{Z}^k, 1)$'s

We return to the determinantal arrangements arising from Cholesky or modified Cholesky type factorizations that we considered in Section 1. We have the following general result for the topology of their complements.

Theorem 3.4. *Each of the determinantal arrangements $\mathcal{E}$ associated to the complex Cholesky-type factorizations in Theorem* 1.4 *and the modified Cholesky-type factorizations in Theorem* 1.5 *have complements which are $K(\mathbb{Z}^k, 1)$'s, where k is the rank of the corresponding solvable group in Table* 1. *Hence, they are homotopy equivalent to k-tori*; *and the Milnor fibers of $(\mathcal{E}, 0)$ are homotopy equivalent to $(k-1)$-tori.*

For the three families of complex symmetric matrices, and modified Cholesky factorizations for general complex $m \times m$ and $(m-1) \times m$ matrices the corresponding determinantal arrangements $\mathcal{E}$ are free divisors with the preceding properties.

Proof. As stated in Theorem 2.5, the corresponding determinantal arrangements for the complex Cholesky and modified Cholesky-type factorizations are the exceptional orbit varieties for the corresponding solvable linear algebraic groups given in Table 1. Then, Theorem 3.1 implies that the complements of the determinantal arrangements are $K(\pi, 1)$'s. Once we have shown that in each case $\pi \simeq \mathbb{Z}^k$, where k is the rank of the corresponding solvable group, it will follow by Corollary 3.3 that the complement is homotopy equivalent to a k-torus and the Milnor fiber, to a $(k-1)$-torus. Furthermore, by Theorem 2.5 for the cases of complex symmetric matrices and the modified Cholesky factorizations, the corresponding exceptional orbit varieties are free divisors with the preceding properties. It remains to show in each case that $\pi_1(V \backslash \mathcal{E}) \simeq \mathbb{Z}^k$ where k is the rank of the corresponding solvable group.

We first consider the determinantal arrangements for the complex Cholesky-type factorization (see Theorem 1.4). In the case of $m \times m$ complex symmetric matrices, the isotropy group for the identity matrix I is $H = (\mathbb{Z}/2\mathbb{Z})^m$ consisting of diagonal matrices with entries ± 1 as the diagonal entries. We claim that the extension of $\mathbb{Z}^m$ by H is again isomorphic to $\mathbb{Z}^m$.

To see this, we consider for a $1 \leqslant j \leqslant m$ a path $\gamma_j(t)$ from $[0, 1]$ to the Borel subgroup B_m of lower triangle matrices. It consists of diagonal matrices $\gamma_j(t) = B_t$ with entries 1 in all positions except for the j-th which is $e^{\pi i t}$. Then $B_t \cdot B_t^T$ is diagonal with all diagonal entries 1 except in the j-th position, where it is $e^{2\pi i t}$. This is a closed path α_j in the complement of the determinantal arrangement. Thus, the corresponding path in the Borel subgroup B_m is a lift of α_j. Also, a lift of $\alpha_j * \alpha_j$ is the path β_j in B_m of diagonal matrices with j-th entry $e^{2\pi i t}$ which defines the j-th generator of $\mathbb{Z}^m$, the fundamental group for B_m, and hence G.

Second, the covering transformation h_j of G corresponding to α_j is given by multiplication by the diagonal matrix H_j, whose (i, i)-entry is 1 if $i \neq j$ and -1 if $i = j$. These generate the group of covering transformations.

Third, because paths $\alpha_i(t)$ and $\alpha_j(t)$ with $i \neq j$ are in different diagonal positions, the path classes $\alpha_j * \alpha_i$ and $\alpha_i * \alpha_j$ are homotopic. Hence, the classes in $\pi_1(V \backslash \mathcal{E}, I)$ defined by $\{\alpha_i(t)\colon\ 1 \leqslant i \leqslant m\}$ commute; they generate the group of covering transformations; and their squares generate $\pi_1(G, I)$. Thus, they generate $\pi_1(V \backslash \mathcal{E}, I)$, which is then a free abelian group generated by the α_i.

Second, for general $m \times m$ complex matrices, by the uniqueness of the complex LU decomposition, the isotropy group is the trivial group. Hence, $\pi \simeq \mathbb{Z}^m$.

Third, for $m \times m$ skew-symmetric matrices with $m = 2\ell$, the isotropy subgroup H of the matrix J in (3) is $H \simeq (\mathbb{Z}/2\mathbb{Z})^\ell$. The generator of the j-th factor is given by the block diagonal matrix with 2×2 blocks which are the identity except for the j-th block which can be the 2×2 diagonal matrix $\pm I$. An analogous argument as for the symmetric case shows that the extension of $\mathbb{Z}^\ell$ by H is again isomorphic to $\mathbb{Z}^\ell$. For $m = 2\ell + 1$, a similar argument likewise shows that $H \simeq (\mathbb{Z}/2\mathbb{Z})^\ell$ and the extension of $\mathbb{Z}^\ell$ by H is again isomorphic to $\mathbb{Z}^\ell$.

Fourth, for both types of modified Cholesky-type factorization the factorization is unique. Hence, in both cases the isotropy subgroups are trivial. Hence, again $\pi \simeq \mathbb{Z}^k$, where k is the rank of the corresponding solvable group in Table 1. It is $k = 2m - 1$ for the $m \times m$ general case and $k = 2m - 2$ for the $(m-1) \times m$ case. $\square$

Example 3.5. We illustrate the preceding with the simplest examples. We consider the lowest dimensional representations of each type. The matrices in each space are given by

$$\text{a)}\ \begin{pmatrix} x & y \\ y & z \end{pmatrix}, \qquad \text{b)}\ \begin{pmatrix} x & y \\ z & w \end{pmatrix}, \qquad \text{c)}\ \begin{pmatrix} x & y & z \\ u & v & w \end{pmatrix}, \qquad \text{d)}\ \begin{pmatrix} 0 & x & y & z \\ -x & 0 & u & v \\ -y & -u & 0 & w \\ -z & -v & -w & 0 \end{pmatrix}. \tag{3.4}$$

Then, Table 2 lists the corresponding representation and the topological type of both the complement and the Milnor fiber of the exceptional orbit varieties. One point to observe is that the equations $xz - y^2$ on $\mathbb{C}^3$, $xw - yz$ on $\mathbb{C}^4$, and $xw - yv + zu$ on $\mathbb{C}^6$ define Morse singularities at 0. Their Milnor fibers are homotopy equivalent to respectively S^2, S^3, and S^5; and the complements are homotopy equivalent to bundles over S^1 with these respective fibers. By adding a plane tangent to an element of each of the cones defined by the equations, the complements and Milnor fibers become homotopy tori.

4. Generators for the cohomology of the Milnor fibers

For the cases of the representations corresponding to both Cholesky-type and modified Cholesky-type factorizations, we will compute explicit generators for the cohomology algebras with complex coefficients of both the complement and the Milnor fiber of the exceptional orbit variety. By Theorem 3.4, it is enough to give a basis for $H^1(\cdot, \mathbb{C})$ for each case.

Table 2
Simplest examples of representations for (modified) Cholesky-type factorizations, with equations defining exceptional orbit varieties $\mathcal{E}$. Listed is the homotopy type of the complement $V \backslash \mathcal{E}$ and the Milnor fiber of $\mathcal{E}$. Note that because the nonlinear group acting on Sk_4 is infinite dimensional, we cannot apply the preceding results to determine the topology of the complement nor the Milnor fiber.

Matrix space	Group	Free/free*	$\mathcal{E}$	$V\backslash\mathcal{E}$	Milnor fiber
Cholesky-type factorization					
Sym_2	B_2	Free	$x(xz - y^2)$	T^2	S^1
$M_{2,2}$	$B_2 \times N_2$	Free*	$x(xw - yz)$	T^2	S^1
Sk_4	D_4	Free*	$x(xw - yv + zu)$	T^2	S^1
Modified Cholesky-type factorization					
$M_{2,2}$	$B_2 \times C_2$	Free	$xy(xw - yz)$	T^3	T^2
$M_{2,3}$	$B_2 \times C_3$	Free	$xy(xv - yu)(yw - zv)$	T^4	T^3
Sk_4	Nonlinear	Free	$xyu(yv - zu)(xw - yv + zu)$		

We will use several facts concerning prehomogeneous spaces due to Sato and Kimura (see e.g. [14] and [21]). Prehomogeneous spaces are representations V of complex algebraic Lie groups G which have open orbits. The exceptional orbit variety $\mathcal{E}$ is again the complement of the open orbit. By Theorem 2.9 in [21], the components of $\mathcal{E}$ which are hypersurfaces have reduced defining equations f_i which are the *basic relative invariants* (they generate the multiplicative group of all relative invariants). f_i is a relative invariant if there is a rational character χ_i of G so that $f_i(g \cdot v) = \chi_i(g) \cdot f_i(v)$ for all $g \in G$ and $v \in V$. The 1-forms $\omega_i = \frac{df_i}{f_i}$ are defined on $V \backslash \mathcal{E}$ and are the pull-backs of $\frac{dz}{z}$ via f_i. Hence, they are closed and define one-dimensional cohomology classes in $H^1(V \backslash \mathcal{E}, \mathbb{C})$. By pulling back the ω_i via the inclusion map of the Milnor fiber $F \hookrightarrow V \backslash \mathcal{E}$ we obtain cohomology classes $\tilde{\omega}_i \in H^1(F, \mathbb{C})$. We have the following description of the cohomology algebras.

Theorem 4.1. *Let $\rho : G \to \mathrm{GL}(V)$ be a (nonreduced) block representation of a solvable linear algebraic group G whose rank is k. Suppose the complement of the exceptional orbit variety $\mathcal{E}$ satisfies $\pi_1(V \backslash \mathcal{E}) \simeq \mathbb{Z}^k$. Then,*

i) *there are k basic relative invariants f_i and $H^1(V \backslash \mathcal{E}, \mathbb{C})$ has the basis ω_i for $i = 1, \ldots, k$. Hence, the cohomology algebra $H^*(V \backslash \mathcal{E}, \mathbb{C})$ is the free exterior algebra on these generators*;
ii) *$H^1(F, \mathbb{C})$ is generated by the $\{\tilde{\omega}_i,\ i = 1, \ldots, k\}$ with a single relation $\sum_{i=1}^k \tilde{\omega}_i = 0$. Hence, $H^*(F, \mathbb{C})$ is the free exterior algebra on any subset of $k - 1$ of the $\tilde{\omega}_i$.*

Because of the explicit generators and relation for the degree 1 cohomology of the Milnor fiber, we can draw the following conclusion.

Corollary 4.2. *For a (nonreduced) block representation as in Theorem 4.1, the Gauss–Manin connection for the exceptional orbit variety $(\mathcal{E}, 0)$ is trivial.*

Proof. By Theorem 4.1, the ω_i restrict to give global sections of the cohomology sheaf $\mathcal{H}^1(U, \mathbb{C}) = H^1(h^{-1}(U), \mathbb{C})$ for $U \subset \mathbb{C}^*$ (and h the reduced defining equation). Hence, the Gauss–Manin connection for the fibration $h : V \backslash \mathcal{E} \to \mathbb{C}^*$ is trivial for each of these elements, as it is for the single relation $\sum \omega_i$. Since their restrictions generate the cohomology of the fiber, the Gauss–Manin connection acts trivially on the entire cohomology. As the inclusion of the Milnor fibration of the exceptional orbit variety into $V \backslash \mathcal{E}$ is a homotopy equivalence of fibrations, the Gauss–Manin connection is also trivial on the Milnor fiber. □

By Theorem 3.4, this theorem applies to all of the representations corresponding to both Cholesky-type and modified Cholesky-type factorizations.

Corollary 4.3. *For each representation $\rho : G \to \mathrm{GL}(V)$ corresponding to a Cholesky or modified Cholesky type factorizations, the conclusions of Theorem 4.1 and Corollary 4.2 apply to the complement of the exceptional orbit variety $\mathcal{E}$, and to the Milnor fiber and Gauss–Manin connection of $(\mathcal{E}, 0)$.*

Example 4.4. For the representation of B_3 on Sym_3, the exceptional orbit variety $\mathcal{E}_3^{sy}$ is defined using coordinates for a generic matrix

$$A = \begin{pmatrix} x & y & z \\ y & w & u \\ z & u & v \end{pmatrix}$$

by

$$x\bigl(xw - y^2\bigr) \cdot \det(A) = 0.$$

By Theorem 4.1 and Corollary 4.3, the complex cohomology of the complement is the exterior algebra

$$H^*\big(Sym_3\backslash\mathcal{E}_3^{sy};\mathbb{C}\big)\simeq\Lambda^*\mathbb{C}\left\langle\frac{dx}{x},\frac{d(xw-y^2)}{(xw-y^2)},\frac{d(\det(A))}{\det(A)}\right\rangle.$$

In addition, the complex cohomology of the Milnor fiber of $\mathcal{E}_3^{sy}$ is isomorphic to the exterior algebra on any two of the preceding generators.

Proof of Theorem 4.1. First, we consider $V\backslash\mathcal{E}$. For $v_0\in V\backslash\mathcal{E}$, the map $\varphi:G\to V\backslash\mathcal{E}$ sending $g\mapsto g\cdot v_0$ is a regular covering space map. Hence, the homomorphism $\varphi_*:\pi_1(G)\to\pi_1(V\backslash\mathcal{E})$ is injective. By the assumption on $V\backslash\mathcal{E}$ and the fact that G is homotopy equivalent to its maximal torus, both are $\mathbb{Z}^k$, where k is the rank of G. Thus, by the Hurewicz theorem and the universal coefficient theorem, we conclude that $\varphi_*:H_1(G,\mathbb{C})\to H_1(V\backslash\mathcal{E},\mathbb{C})$ is injective, and both groups are isomorphic to $\mathbb{C}^k$; hence, φ_* is an isomorphism. Thus, also $\varphi^*:H^1(V\backslash\mathcal{E},\mathbb{C})\simeq H^1(G,\mathbb{C})$. Hence, if $\{f_1,\dots,f_m\}$ denotes the set of basic relative invariants, we shall show that $m=k$ and that $\varphi^*(\omega_i)$ for $i=1,\dots,k$ form a set of generators for $H^1(G,\mathbb{C})$.

Consider one f_i with its corresponding character χ_i. Consider a one-parameter subgroup $\exp(tw)$ for $w\in\mathfrak{t}$, the Lie algebra of a maximal torus T of G. Then, for any $v\in V\backslash\mathcal{E}$,

$$f_i\big(\exp(tw)\cdot v\big)=\chi_i\big(\exp(tw)\big)f_i(v).\tag{4.1}$$

Since $\exp:\mathfrak{t}\to T$ is a Lie group homomorphism, so is $\chi_i\circ\exp$. Thus, if $\{w_1,\dots,w_k\}$ is a basis for $\mathfrak{t}$, then χ_i has the following form on T,

$$\chi_i\Big(\exp\Big(t\Big(\sum z_\ell w_\ell\Big)\Big)\Big)=\exp\Big(t\Big(\sum\lambda_\ell^{(i)}z_\ell\Big)\Big).\tag{4.2}$$

Then, for $w=\sum z_\ell w_\ell$, substituting (4.2) into (4.1), and differentiating with respect to t, we obtain

$$\frac{\partial f_i(\exp(tw)\cdot v)}{\partial t}=\frac{\partial\exp(t(\sum\lambda_\ell^{(i)}z_\ell))}{\partial t}f_i(v).\tag{4.3}$$

The LHS of (4.3) computes $df_i(\xi_w(\exp(tw)\cdot v))$, where ξ_w is the representation vector field associated to w. Thus, we obtain

$$df_i\big(\xi_w\big(\exp(tw)\cdot v\big)\big)=\Big(\sum\lambda_\ell^{(i)}z_\ell\Big)\cdot\exp\Big(t\Big(\sum\lambda_\ell^{(i)}z_\ell\Big)\Big)f_i(v)\tag{4.4}$$

or (4.4) can be rewritten as

$$\frac{1}{f_i}\cdot df_i(\xi_w)\big(\exp(tw)\cdot v\big)=\sum\lambda_\ell^{(i)}z_\ell.\tag{4.5}$$

Hence,

$$\omega_i(\xi_w)\big(\exp(tw)\cdot v\big)=\sum_{\ell=1}^{k}\lambda_\ell^{(i)}z_\ell.\tag{4.6}$$

By the Lie–Kolchin theorem, we may suppose that G is a subgroup of a Borel subgroup B_r of some $\mathrm{GL}_r(\mathbb{C})$, and the maximal torus T is a subgroup of the torus $T^r=(\mathbb{C}^*)^r$. Thus, we may choose our generators $w_j=2\pi iu_j$ with $u_j\in\mathbb{C}^r$ so that the $\gamma_j(t)=\exp(tw_j)=\exp(2\pi itu_j)$, $0\leqslant t\leqslant 1$, each parametrizes an $S^1\subset T$; and the corresponding set of fundamental classes for $j=1,\dots,k=\operatorname{rank}(G)$ generates $H_1(T,\mathbb{Z})$. Since $T\hookrightarrow G$ is a homotopy equivalence, they also generate $H_1(G,\mathbb{Z})$. Furthermore, their images in $H_1(G,\mathbb{C})$ form a set of generators which are mapped by φ_* to a set of generators for $H_1(V\backslash\mathcal{E},\mathbb{C})$. These are defined by $\delta_i(t)=\gamma_i(t)\cdot v_0$.

Next, we evaluate ω_j on them.

$$\int_{\delta_j}\omega_i=\int_0^1\omega_i\big(\delta_j'\big)\big(\gamma_j(t)\cdot v_0\big)\,dt=\int_0^1\omega_i(\xi_{w_j})\big(\exp(tw_j)\cdot v_0\big)\,dt.\tag{4.7}$$

Applying (4.6), keeping in mind that for w_j, $z_\ell=0$ for $\ell\neq j$, we obtain

$$\int_0^1\sum\lambda_\ell^{(i)}z_\ell\,dt=\lambda_j^{(i)}.$$

Hence,

$$\int_{\delta_j} \omega_i = \lambda_j^{(i)}. \tag{4.8}$$

As we vary over the set of basic relative invariants $\{f_1, \ldots, f_m\}$, we obtain an $m \times k$ matrix $\Lambda = (\lambda_j^{(i)})$ which by (4.2) yields for the characters $\{\chi_i \circ \exp\colon\ i = 1, \ldots, m\}$, a representation of the set of corresponding infinitesimal characters on $\mathfrak{t}$ with respect to the dual basis for $\{w_1, \ldots, w_k\}$.

First, by the theory of prehomogeneous vector spaces, [14, Theorem 2.9], the set of characters for the basic relative invariants are multiplicatively independent in the character group $X(G) \simeq X(G/[G,G]) \simeq X(T)$, for T a maximal torus. This is a free abelian group of rank $k = \mathrm{rank}(G) = \mathrm{rank}(T)$. Hence, $m \leqslant k$.

Second, by [14, Proposition 2.12], the characters $\{\chi_i\colon\ i = 1, \ldots, m\}$ generate $X(G_1) \simeq X(T/H)$, where in our case, G_1 is the quotient of G by the group generated by the unipotent radical N of G and the isotropy subgroup of an element v_0 in the open orbit. Here H denotes the image of the isotropy subgroup in $G/N \simeq T$. As a consequence of G being solvable, there is a torus T in G so composition with projection onto G/N is an isomorphism. Hence, via this isomorphism, we may assume $H \subset T$. As H is finite, T/H is a torus of the same dimension and the map $T \to T/H$ induces an isomorphism on the corresponding Lie algebras. Thus, $\{\chi_i\colon\ i = 1, \ldots, m\}$ generate $X(T/H)$, an abelian group of rank k, so $m \geqslant k$.

Hence, $m = k$ and the $\{\chi_i\}$ are algebraically independent in $X(T/H)$, which implies the corresponding infinitesimal characters on $\mathfrak{t}$ are linearly independent. This is equivalent to Λ being nonsingular.

Hence, by (4.8), we conclude that the $\{\omega_i\}$ form a set of generators for $H^1(V \backslash \mathcal{E}, \mathbb{C})$.

Lastly, it remains to show that if F is the Milnor fiber of $(\mathcal{E}, 0)$, then the $\{\tilde{\omega}_i\}$ form a spanning set for $H^1(F, \mathbb{C})$ with single relation $\sum_{i=1}^k \tilde{\omega}_i = 0$. By our earlier arguments, if h is a reduced defining equation for $\mathcal{E}$, we may use $F = h^{-1}(1)$. By assumption if f_i, $i = 1, \ldots, k$, are the basic relative invariants, then $h = \prod_{i=1}^k f_i$ is a reduced defining equation for $\mathcal{E}$.

We let $i : F \hookrightarrow V \backslash \mathcal{E}$ denote the inclusion, so $\tilde{\omega}_i = i^*(\omega_i)$. As $i_* : \pi_1(F) \to \pi_1(V \backslash \mathcal{E})$ is the inclusion $\mathbb{Z}^{k-1} \hookrightarrow \mathbb{Z}^k$ where $k =$ rank of G, by the Hurewicz theorem and universal coefficient theorem, $i^* : H^1(V \backslash \mathcal{E}, \mathbb{C}) \to H^1(F, \mathbb{C})$ is a surjective map $\mathbb{C}^k \to \mathbb{C}^{k-1}$. We need only identify the one-dimensional kernel. Since $h = \prod_{i=1}^k f_i$ and F is defined by $h = 1$, we can differentiate the equation on F to obtain

$$\sum_{i=1}^{k} \frac{df_i}{f_i}\bigg|_F = 0, \tag{4.9}$$

i.e.

$$\sum_{i=1}^{k} \tilde{\omega}_i = \sum_{i=1}^{k} i^* \omega_i = 0.$$

As this is a one-dimensional subspace of $H^1(V \backslash \mathcal{E}, \mathbb{C})$ in the kernel of i^*, it must span the entire kernel as claimed. Since we know F is homotopy equivalent to a $(k-1)$-torus, $H^1(F, \mathbb{C})$ is an exterior algebra on $k-1$ generators and these may be chosen to be any $k-1$ of the $\{\tilde{\omega}_i\}$. $\square$

5. Cholesky-type factorizations for parametrized families

We point out a simple consequence of the theorems for the question of when, for a continuous or smooth family of complex matrices, (modified) Cholesky-type factorization can be continuously or smoothly applied to the family of matrices. To consider all cases together, we view (modified) Cholesky-type factorization as giving a factorization $A = B \cdot K \cdot C$, for appropriate K, B and C, with possible relations between B and C. For example, for complex symmetric matrices, $K = I$, and B is lower triangular with $C = B^T$.

In each case, for a continuous or smooth family $A_s, s \in X$, we seek continuous (or smooth) families B_s and C_s so that $A_s = B_s \cdot K \cdot C_s$ for all $s \in X$. While it may be possible for each individual A_s to have a (modified) Cholesky-type factorization, it may not be possible to do so in a continuous or smooth manner.

5.1. Parametrized families of real matrices

First for all three real cases of Cholesky-type factorization we have a unique representation. Hence, the orbit map in the real case is a diffeomorphism, so we may obtain a continuous or smooth factorization for the family by composing with the inverse.

However, in this case the open orbit has a much simpler structure. The real solvable groups are not connected, so the open orbits are a union of connected components, each of which is diffeomorphic to the connected component of the group. The groups have connected components which have as a maximal torus a "split torus" which is isomorphic to $(\mathbb{R}_+)^k \simeq \mathbb{R}^k$, for appropriate k. As the connected component is again an extension of this torus by a real nilpotent group, which is again Euclidean, we conclude that the connected components are diffeomorphic to a Euclidean space, and hence contractible, and the orbit map is a diffeomorphism on each component.

Hence, in addition to the continuity or smoothness of factorizations in families more is true. If (X, Y) is a CW-pair and there is a continuous or smooth family A_s, $s \in Y$, with Cholesky-type factorization for a given type, the A_s can be extended to a continuous or smooth family on X which still has continuous, respectively smooth Cholesky factorization of the same type.

5.2. Parametrized families of complex matrices

By contrast with the real case, as a result of the structure of the complement to the exceptional orbit varieties for both Cholesky and modified Cholesky-type factorizations, the answer is different.

First, in the case of general $m \times m$ or $(m-1) \times m$ matrices, the LU or modified LU factorizations are unique. Hence, the orbit maps $G \to V \backslash \mathcal{E}$ are diffeomorphisms. Hence, families can be continuously or smoothly factored. However, for both symmetric and skew-symmetric matrices, there is finite isotropy so the orbit map $G \to V \backslash \mathcal{E}$ is a covering space and the continuous or smooth factorization involves lifting a map into $V \backslash \mathcal{E}$ up to G. There are well-known criteria for such a lifting from covering space theory. We show that these conditions can be restated in terms of the cohomology classes ω_i (for each case), so that they define obstructions to such a lifting for either symmetric or skew-symmetric Cholesky-type factorization.

First we can use the ω_j to define integral cohomology classes. If γ is a smooth closed loop in $V \backslash \mathcal{E}$, then since $\omega_j = f_j^*(\frac{dz}{z})$

$$\int_{\gamma} \omega_j = \int_{f_j \circ \gamma} \frac{dz}{z} = 2\pi i n$$

where n is the winding number of $f_j \circ \gamma$ about 0. Thus, the integral of $\frac{1}{2\pi i}\omega_j$ over any smooth closed loop in $V \backslash \mathcal{E}$ is an integer, which implies that it defines an integral cohomology class in $H^1(V \backslash \mathcal{E}, \mathbb{Z})$. In either the symmetric or skew-symmetric case, we can define a homomorphism

$$\begin{aligned} \boldsymbol{\omega} : H_1(V \backslash \mathcal{E}, \mathbb{Z}) &\to \mathbb{Z}^k \\ u &\mapsto \left(\left\langle \frac{1}{2\pi i}\omega_1, u \right\rangle, \ldots, \left\langle \frac{1}{2\pi i}\omega_k, u \right\rangle \right), \end{aligned} \tag{5.1}$$

where $\langle \cdot, \cdot \rangle$ is the Kronecker product. Then, we let $\boldsymbol{\omega}_2$ be the composition of $\boldsymbol{\omega}$ with the projection $\mathbb{Z}^k \to (\mathbb{Z}/2\mathbb{Z})^k$. As a consequence of the results in the preceding sections, we have the following corollary.

Corollary 5.1. *Suppose X is a locally path-connected space and that $\varphi : X \to M$ defines a continuous, respectively smooth (with X a manifold), mapping to a space of matrices so that for each $s \in X$, $A_s = \varphi(s)$ has a (modified) Cholesky-type factorization for any fixed one of the types considered in Section 1. If we are either*

(1) *in the case of general matrices with either Cholesky or modified Cholesky-type factorization; or*
(2) *in the symmetric or skew-symmetric cases, and the obstruction $\boldsymbol{\omega}_2 \circ \varphi_* = 0$,*

then there is a continuous, respectively smooth, (modified) Cholesky-type factorization $A_s = B_s \cdot K \cdot C_s$ defined for all $s \in X$.

Proof. By an earlier remark, the conclusion already follows for (1). It is enough to consider case (2).

We may denote the group of matrices acting on M by G, and let $\mathcal{U}$ denote the open orbit, which is complement of the exceptional orbit variety $\mathcal{E}$. Then, by assumption, $\varphi : X \to \mathcal{U}$. By composing φ with an element of G, we may suppose $\varphi(s_0) = K$. Also, we may consider each path component of X separately, so we may as well assume X is path connected.

By Theorem 3.4 and the proof of Theorem 3.1, for each type of Cholesky-type factorization, the map $p : G \to \mathcal{U}$ sending $g \mapsto g \cdot v_0$ is a smooth finite covering space (where we let $v_0 = K$). Furthermore, by the proof of Theorem 3.4 for the symmetric or skew-symmetric cases, $p_* : \pi_1(G, 1) \to \pi_1(\mathcal{U}, v_0)$ is the inclusion $\mathbb{Z}^k \hookrightarrow \mathbb{Z}^k$ with image $(2\mathbb{Z})^k$. Since X is locally path-connected and path-connected, by covering space theory, there is a lift of φ to $\tilde{\varphi} : X \to G$ (smooth if φ is smooth) with $\tilde{\varphi}(s_0) = 1$, if and only if $\varphi_*(\pi_1(X, s_0)) \subset p_*(\pi_1(G, 1))$. As $\pi_1(\mathcal{U}, v_0) \simeq \mathbb{Z}^k$ is abelian, by the Hurewicz theorem, this is equivalent to $\varphi_*(H_1(X, s_0)) \subset (2\mathbb{Z})^k$. However, this holds exactly when $\boldsymbol{\omega}_2 \circ \varphi_* = 0$.

Then, by the definition of the covering map p, the lift gives the continuous, resp. smooth, Cholesky-type factorization for all $s \in X$. □

Remark 5.2. The obstruction in Corollary 5.1 will always vanish if e.g. $H_1(X, \mathbb{Z})$ is a torsion group.

From the structure of the complement $V \backslash \mathcal{E}$ being homotopy equivalent to a torus for any of the cases of Cholesky or modified Cholesky-type factorization, we can also give a sufficient condition for the extension problem. For a CW-pair (X, Y), a sufficient condition for the extension of a continuous or smooth family A_s on Y, which has a continuous or smooth Cholesky factorization of given type, to a continuous or smooth family on X having the Cholesky factorization of the same type is that (X, Y) is 1-connected.

References

[1] P. Benner, R. Byers, H. Fassbender, V. Mehemann, D. Watkins, Cholesky-like factorizations for skew-symmetric matrices, Electron. Trans. Numer. Anal. 11 (2000) 85–93.
[2] A. Borel, Linear Algebraic Groups, second edition, Grad. Texts in Math., vol. 126, Springer-Verlag Publ., 1991.
[3] E.V. Brieskorn, Sur les Groupes de Tresses (aprés Arnold), in: Séminaire Bourbaki 1971/1972, in: Springer Lecture Notes in Math., vol. 317, 1973.
[4] E.V. Brieskorn, Die Fundamentalgruppe des Raumes der Regulären Orbits einer endlichen complexer Spiegelungsgruppe, Invent. Math. 12 (1971) 57–61.
[5] R.O. Buchweitz, D. Mond, Linear free divisors and quiver representations, in: Singularities and Computer Algebra, in: London Math. Soc. Lecture Note Ser., vol. 324, Cambridge Univ. Press, 2006, pp. 41–77.
[6] J. Damon, On the legacy of free divisors II: Free* divisors and complete intersections, in: Special Issue in Honor of V.I. Arnol'd, Mosc. Math. J. 3 (2) (2003) 361–395.
[7] J. Damon, D. Mond, $\mathcal{A}$-codimension and the vanishing topology of discriminants, Invent. Math. 106 (1991) 217–242.
[8] J. Damon, B. Pike, Solvable groups, free divisors and nonisolated matrix singularities I: Towers of free divisors, submitted for publication.
[9] J. Damon, B. Pike, Solvable groups, free divisors and nonisolated matrix singularities II: Vanishing topology, preliminary version.
[10] P. Deligne, Les Immenbles des Groupes de Tresses Généralisés, Invent. Math. 17 (1972) 273–302.
[11] J.W. Demmel, Applied Numerical Linear Algebra, SIAM Publ., 1997.
[12] M. Granger, D. Mond, A. Nieto-Reyes, M. Schulze, Linear free divisors and the global logarithmic comparison theorem, Ann. Inst. Fourier 59 (2009) 811–850.
[13] A.W. Knapp, Lie Groups Beyond an Introduction, second edition, Progr. Math., vol. 140, Birkhäuser, Boston, 2002.
[14] T. Kimura, Introduction to Prehomogeneous Vector Spaces, Transl. Math. Monogr., vol. 215, Amer. Math. Soc., 2003.
[15] H. Knörrer, Zum $K(\pi, 1)$ Problem für isolierte Singularitäten von vollständigen Durchschnitten, Compos. Math. 45 (1982) 333–340.
[16] E.J.N. Looijenga, Isolated Singular Points on Complete Intersections, London Math. Soc. Lecture Note Ser., vol. 77, Cambridge Univ. Press, 1984.
[17] P. Orlik, L. Solomon, Discriminants in the invariant theory of reflection groups, Nagoya Math. J. 109 (1988) 23–45.
[18] P. Orlik, H. Terao, Arrangements of Hyperplanes, Grundlehren Math. Wiss., vol. 300, Springer-Verlag, 1992.
[19] B. Pike, PhD thesis, Dept. of Mathematics, University of North Carolina, 2010.
[20] K. Saito, Theory of logarithmic differential forms and logarithmic vector fields, J. Fac. Sci. Univ. Tokyo Sect. Math. 27 (1980) 265–291.
[21] M. Sato, T. Kimura, A classification of irreducible prehomogeneous vector spaces and their relative invariants, Nagoya Math. J. 65 (1977) 1–155.